

On a theorem of Eilenberg in simplicial Bredon-Illman cohomology with local coefficients

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Abstract

We prove simplicial version of a classical theorem of Eilenberg in the equivariant context and give an alternative description of the simplicial version of Bredon-Illman cohomology with local coefficients, as introduced in [16], to derive a spectral sequence.

Keywords: Simplicial sets, Kan fibration, universal cover, local coefficients, group action, equivariant cohomology.

1 Introduction

For spaces with group actions the analogue of cohomology with local coefficients is the Bredon-Illman cohomology with local coefficients, as introduced in [13]. This is based on the notion of fundamental groupoid of a space equipped with a group action. A classical theorem of Eilenberg states that cohomology with local coefficients of a space can be described by the cohomology of an invariant subcomplex of the cochain complex of its universal cover, where the universal cover is equipped with the action of the fundamental group of the base space. An equivariant analogue of this result was proved in [13]. Recently, in [16], we introduced equivariant simplicial cohomology with local coefficients, which is the simplicial version of Bredon-Illman cohomology with local coefficients and proved a classification theorem. The corresponding non-equivariant result was proved in [9], [7], [3]. In this paper we derive Eilenberg's theorem for equivariant simplicial cohomology with local coefficients. This is based on the notion of universal covering complexes of one vertex Kan complexes [8]. In equivariant context, the role of the universal cover is played by a contravariant functor from the category of canonical orbits to the category of one vertex Kan complexes. Finally, we give an alternative description of equivariant simplicial cohomology with local coefficients via the notion of cohomology of a small category following [15] and use it to derive a spectral sequence.

The paper is organized as follows. In Section 2, we recall some standard results and fix notations. The notion of equivariant local coefficients of a simplicial set equipped with a simplicial group action is based on fundamental groupoid. In Section 3, we recall these concepts and the definition of simplicial version of Bredon-Illman cohomology with local coefficient

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2 Preliminaries on G -simplicial sets

In this section we set up our notations and recall some standard facts [11].

Let Δ be the category whose objects are ordered sets

$$[n] = \{0 < 1 < \cdots < n\}, \quad n \geq 0,$$

and morphisms are non-decreasing maps $f : [n] \rightarrow [m]$. There are some distinguished morphisms $d^i : [n-1] \rightarrow [n]$, $0 \leq i \leq n$, called cofaces and $s^i : [n+1] \rightarrow [n]$, $0 \leq i \leq n$, called codegeneracies, defined as follows:

$$\begin{aligned} d^i(j) &= j, \quad j < i \quad \text{and} \quad d^i(j) = j+1, \quad j \geq i, \quad (n > 0, \quad 0 \leq i \leq n); \\ s^i(j) &= j, \quad j \leq i, \quad \text{and} \quad s^i(j) = j-1, \quad j > i, \quad (n \geq 0, \quad 0 \leq i \leq n). \end{aligned}$$

These maps satisfy the standard cosimplicial relations.

A simplicial object X in a category \mathcal{C} is a contravariant functor $X : \Delta \rightarrow \mathcal{C}$. Equivalently, a simplicial object is a sequence $\{X_n\}_{n \geq 0}$ of objects of \mathcal{C} , together with \mathcal{C} -morphisms $\partial_i : X_n \rightarrow X_{n-1}$ and $s_i : X_n \rightarrow X_{n+1}$, $0 \leq i \leq n$, verifying the following simplicial identities:

$$\begin{aligned} \partial_i \partial_j &= \partial_{j-1} \partial_i, \quad \partial_i s_j = s_{j-1} \partial_i, \quad \text{if } i < j, \\ \partial_j s_j &= id = \partial_{j+1} s_j, \\ \partial_i s_j &= s_j \partial_{i-1}, \quad i > j+1; \quad s_i s_j = s_{j+1} s_i, \quad i \leq j. \end{aligned}$$

A simplicial map $f : X \rightarrow Y$ between two simplicial objects in a category \mathcal{C} , is a collection of \mathcal{C} -morphisms $f_n : X_n \rightarrow Y_n$, $n \geq 0$, commuting with ∂_i and s_i .

In particular, a simplicial set is a simplicial object in the category of sets. Throughout \mathcal{S} will denote the category of simplicial sets and simplicial maps.

For any n -simplex $x \in X_n$, in a simplicial set X , we shall use the notation $\partial_{(i_1, i_2, \dots, i_r)} x$ to denote the simplex $\partial_{i_1} \partial_{i_2} \cdots \partial_{i_r} x$ obtained by applying the successive face maps $\partial_{i_{r-k}}$ on x , where $0 \leq i_{r-k} \leq n-k$, $0 \leq k \leq r-1$.

Recall that the simplicial set $\Delta[n]$, $n \geq 0$, is defined as follows. The set of q -simplices is

$$\Delta[n]_q = \{(a_0, a_1, \dots, a_q); \text{ where } a_i \in \mathbb{Z}, \quad 0 \leq a_0 \leq a_1 \leq \cdots \leq a_q \leq n\}.$$

The face and degeneracy maps are defined by

$$\partial_i(a_0, \dots, a_q) = (a_0, \dots, a_{i-1}, a_{i+1}, \dots, a_q), \quad s_j(a_0, \dots, a_q) = (a_0, \dots, a_i, a_i, \dots, a_q).$$

Alternatively, the set of k -simplices can be viewed as the contravariant functor

$$\Delta[n]([k]) = \text{Hom}_\Delta([k], [n]),$$

the set of Δ -morphisms from $[k]$ to $[n]$. The only non-degenerate n -simplex is $id : [n] \rightarrow [n]$ and is denoted by Δ_n . In the earlier notation, it is simply, $\Delta_n = (0, 1, \dots, n)$.

It is well known that if X is a simplicial set, then for any n -simplex $x \in X_n$ there is a unique simplicial map $\bar{x} : \Delta[n] \rightarrow X$ with $\bar{x}(\Delta_n) = x$. Often by an n -simplex in a simplicial set X we shall mean either an element $x \in X_n$ or the corresponding simplicial map \bar{x} .

We have simplicial maps $\delta_i : \Delta[n-1] \rightarrow \Delta[n]$ and $\sigma_i : \Delta[n+1] \rightarrow \Delta[n]$ for $0 \leq i \leq n$ defined by $\delta_i(\Delta_{n-1}) = \partial_i(\Delta_n)$ and $\sigma_i(\Delta_{n+1}) = s_i(\Delta_n)$. The boundary subcomplex $\partial\Delta[n]$ of $\Delta[n]$ is defined as the smallest subcomplex of $\Delta[n]$ containing the faces $\partial_i \Delta_n$, $i = 0, 1, \dots, n$.

Definition 2.1. Let G be a discrete group. A G -simplicial set is a simplicial object in the category of G -sets. More precisely, a G -simplicial set is a simplicial set $\{X_n; \partial_i, s_i, 0 \leq i \leq n\}_{n \geq 0}$ such that each X_n is a G -set and the face maps $\partial_i : X_n \rightarrow X_{n-1}$ and the degeneracy maps $s_i : X_n \rightarrow X_{n+1}$ commute with the G -action. A map between G -simplicial sets is a simplicial map which commutes with the G -action.

Definition 2.2. A G -simplicial set X is called G -connected if each fixed point simplicial set X^H , $H \subseteq G$, is connected.

Definition 2.3. Two G -maps $f, g : K \rightarrow L$ between two G -simplicial sets are G -homotopic if there exists a G -map $F : K \times \Delta[1] \rightarrow L$ such that

$$F \circ (id \times \delta_1) = f, \quad F \circ (id \times \delta_0) = g.$$

The map F is called a G -homotopy from f to g and we write $F : f \simeq_G g$. If $i : K' \subseteq K$ is an inclusion of subcomplex and f, g agree on K' then we say that f is G -homotopic to g relative to K' if there exists a G homotopy $F : f \simeq_G g$ such that $F \circ (i \times id) = \alpha \circ pr_1$, where $\alpha = f|_{K'} = g|_{K'}$ and $pr_1 : K' \times \Delta[1] \rightarrow K'$ is the projection onto the first factor. In this case we write $F : f \simeq_G g(\text{rel } K')$.

Definition 2.4. A G -simplicial set is a G -Kan complex if for every subgroup $H \subseteq G$ the fixed point simplicial set X^H is a Kan complex.

Remark 2.5. Recall ([1], [5]) that the category GS of G -simplicial sets and G -simplicial maps between G -simplicial sets has a closed model structure [17], where the fibrant objects are the G -Kan complexes and cofibrant objects are the G -simplicial sets. From this it follows that G -homotopy on the set of G -simplicial maps $K \rightarrow L$ is an equivalence relation, for every G -simplicial set K and G -Kan complex L . More generally, relative G -homotopy is an equivalence relation if the target is a G -Kan complex.

We consider $G/H \times \Delta[n]$ as a simplicial set where $(G/H \times \Delta[n])_q = G/H \times (\Delta[n])_q$ with face and degeneracy maps as $id \times \partial_i$ and $id \times s_i$. Note that the group G acts on G/H by left translation. With this G -action on the first factor and trivial action on the second factor $G/H \times \Delta[n]$ is a G -simplicial set.

A G -simplicial map $\sigma : G/H \times \Delta[n] \rightarrow X$ is called an equivariant n -simplex of type H in X .

Remark 2.6. We remark that for a G -simplicial set X , the set of equivariant n -simplices in X is in bijective correspondence with n -simplices of X^H . For an equivariant n -simplex σ , the corresponding n -simplex is $\sigma' = \sigma(eH, \Delta_n)$. The simplicial map $\Delta[n] \rightarrow X^H$, $\Delta_n \mapsto \sigma'$ will be denoted by $\bar{\sigma}$.

We shall call σ degenerate or non-degenerate according as the n -simplex $\sigma' \in X_n^H$ is degenerate or non-degenerate.

Recall that the category of canonical orbits, denoted by O_G , is a category whose objects are cosets G/H , as H runs over the all subgroups of G . A morphism from G/H to G/K is a G -map. Recall that such a morphism determines and is determined by a subconjugacy relation $g^{-1}Hg \subseteq K$ and is given by $\hat{g}(eH) = gK$. We denote this morphism by \hat{g} [2].

Definition 2.7. A contravariant functor from O_G to \mathcal{S} is called an O_G -simplicial set. A map between O_G -simplicial sets is a natural transformation of functors.

We shall denote the category of O_G -simplicial sets by $O_G\mathcal{S}$.

The notion of O_G -groups or O_G -abelian groups has the obvious meaning replacing \mathcal{S} by \mathcal{Grp} or \mathcal{Ab} .

For a G -simplicial set X , with a G -fixed 0-simplex v , we have an O_G -group $\underline{\pi}X$ defined as follows. For any subgroup H of G ,

$$\underline{\pi}X(G/H) := \pi_1(X^H, v)$$

and for a morphism $\hat{g} : G/H \rightarrow G/K$, $g^{-1}Hg \subseteq K$, $\underline{\pi}X(\hat{g})$ is the homomorphism in fundamental groups induced by the simplicial map $g : X^K \rightarrow X^H$.

Definition 2.8. An O_G -group $\underline{\pi}$ is said to act on an O_G -simplicial set (group or abelian group) \underline{X} if for every subgroup $H \subseteq G$, $\underline{\pi}(G/H)$ acts on $\underline{X}(G/H)$ and this action is natural with respect to maps of O_G . Thus if

$$\phi(G/H) : \underline{\pi}(G/H) \times \underline{X}(G/H) \rightarrow \underline{X}(G/H)$$

denotes the action of $\underline{\pi}(G/H)$ on $\underline{X}(G/H)$ then for each subconjugacy relation $g^{-1}Hg \subseteq K$,

$$\phi(G/H) \circ (\underline{\pi}(\hat{g}) \times \underline{X}(\hat{g})) = \underline{X}(\hat{g}) \circ \phi(G/K).$$

3 Simplicial Bredon-Illman Cohomology with local coefficients

In this section we recall [16] the notion of fundamental groupoid of a G -simplicial set X , the notion of equivariant local coefficients on X and the definition of simplicial Bredon-Illman cohomology with local coefficients.

We begin with the notion of fundamental groupoid. Recall [7] that the fundamental groupoid πX of a Kan complex X is a category having as objects all 0-simplexes of X and a morphism $x \rightarrow y$ in πX is a homotopy class of 1-simplices $\omega : \Delta[1] \rightarrow X$ rel $\partial\Delta[1]$ such that $\omega \circ \delta_0 = \overline{y}$, $\omega \circ \delta_1 = \overline{x}$. If ω_2 represents an arrow from x to y and ω_0 represents an arrow from y to z , then their composite $[\omega_0] \circ [\omega_2]$ is represented by $\Omega \circ \delta_1$, where the simplicial map $\Omega : \Delta[2] \rightarrow X$ corresponds to a 2-simplex, which is determined by the compatible pair (ω'_0, ω'_2) . For a simplicial set X the notion of fundamental groupoid is defined via the geometric realization and the total singular functor.

Suppose x_H and y_K are equivariant 0-simplices of type H and K , respectively, and $\hat{g} : G/H \rightarrow G/K$ is a morphism in O_G , given by a subconjugacy relation $g^{-1}Hg \subseteq K$, $g \in G$, so that $\hat{g}(eH) = gK$. Moreover suppose that we have an equivariant 1-simplex $\phi : G/H \times \Delta[1] \rightarrow X$ of type H such that

$$\phi \circ (id \times \delta_1) = x_H, \quad \phi \circ (id \times \delta_0) = y_K \circ (\hat{g} \times id).$$

Then, in particular, ϕ' is a 1-simplex in X^H such that $\partial_1\phi' = x'_H$ and $\partial_0\phi' = gy'_K$, notations are as in Remark 2.6. Observe that the 0-simplex gy'_K in X^H corresponds to the composition

$$G/H \times \Delta[0] \xrightarrow{\hat{g} \times id} G/K \times \Delta[0] \xrightarrow{y_K} X$$

and ϕ is a G -homotopy $x_H \simeq_G y_K \circ (\hat{g} \times id)$.

Definition 3.1. Let X be a G -Kan complex. The fundamental groupoid ΠX is a category with objects equivariant 0-simplices

$$x_H : G/H \times \Delta[0] \rightarrow X$$

of type H , as H varies over all subgroups of G . Given two objects x_H and y_K in ΠX , a morphism from $x_H \rightarrow y_K$ is defined as follows. Consider the set of all pairs (\hat{g}, ϕ) where $\hat{g} : G/H \rightarrow G/K$ is a morphism in O_G , given by a subconjugacy relation $g^{-1}Hg \subseteq K$, $g \in G$ so that $\hat{g}(eH) = gK$ and $\phi : G/H \times \Delta[1] \rightarrow X$ is an equivariant 1-simplex such that

$$\phi \circ (id \times \delta_1) = x_H, \quad \phi \circ (id \times \delta_0) = y_K \circ (\hat{g} \times id).$$

The set of morphisms in ΠX from x_H to y_K is a quotient of the set of pairs mentioned above by an equivalence relation \sim , where $(\hat{g}_1, \phi_1) \sim (\hat{g}_2, \phi_2)$ if and only if $g_1 = g_2 = g$ (say) and there exists a G -homotopy $\Phi : G/H \times \Delta[1] \times \Delta[1] \rightarrow X$ of G -homotopies such that $\Phi : \phi_1 \simeq_G \phi_2$ (rel $G/H \times \partial\Delta[1]$). Since X is a G -Kan complex, by Remark 2.5, \sim is an equivalence relation. We denote the equivalence class of (\hat{g}, ϕ) by $[\hat{g}, \phi]$. The set of equivalence classes is the set of morphisms in ΠX from x_H to y_K .

The composition of morphisms in ΠX is defined as follows. Given two morphisms

$$x_H \xrightarrow{[\hat{g}_1, \phi_1]} y_K \xrightarrow{[\hat{g}_2, \phi_2]} z_L$$

their composition $[\hat{g}_2, \phi_2] \circ [\hat{g}_1, \phi_1]$ is $[\widehat{g_1 g_2}, \psi] : x_H \rightarrow z_L$, where the first factor is the composition

$$G/H \xrightarrow{\hat{g}_1} G/K \xrightarrow{\hat{g}_2} G/L$$

and $\psi : G/H \times \Delta[1] \rightarrow X$ is an equivariant 1-simplex of type H as described below. Let x be a 2-simplex in the Kan complex X^H determined by the compatible pair of 1-simplices $(g_1\phi'_2, \phi'_1)$ so that $\partial_0 x = g_1\phi'_2$ and $\partial_2 x = \phi'_1$. Then ψ is given by $\psi(eH, \Delta_1) = \partial_1 x$.

It is proved in [16] that the composition is well defined.

For a version of fundamental groupoid of a G -space, we refer [10] and [13].

Observe that if X is a G -simplicial set then $S[X]$ is a G -Kan complex, where for any space Y , SY denotes the total singular complex and for any simplicial set X , $|X|$ denotes the geometric realization of X .

Definition 3.2. For any G -simplicial set X , we define the fundamental groupoid ΠX of X by $\Pi X := \Pi S[X]$.

Remark 3.3. If G is trivial then ΠX reduces to fundamental groupoid πX of a simplicial set X . Again, for a fixed H , the objects x_H together with the morphisms $x_H \rightarrow y_H$ with identity in the first factor, constitute a subcategory of ΠX which is precisely the fundamental groupoid πX^H of X^H . Moreover, a morphism $[\hat{g}, \phi]$ from x_H to y_K , corresponds to the morphism $[\bar{\phi}]$ in the fundamental groupoid πX^H of X^H from x'_H to ay'_K , where $\bar{\phi}$ is as in 2.6. Suppose ξ is a morphism in πX^H from x to y given by a homotopy class $[\bar{\omega}]$, where $\bar{\omega} : \Delta[1] \rightarrow X^H$ represents the 1-simplex in X^H from x to y . Let x_H and y_H be the objects in πX^H defined respectively by

$$x_H(eH, \Delta_0) = x, \quad y_H(eH, \Delta_0) = y.$$

Then we have a morphism $[id, \omega] : x_H \longrightarrow y_H$ in ΠX , where $\omega(eH, \Delta_1) = \overline{\omega}(\Delta_1)$. We shall denote this morphism corresponding to ξ by $b\xi$.

Definition 3.4. An equivariant local coefficients on a G -simplicial set X is a contravariant functor from ΠX to the category $\mathcal{A}b$ of abelian groups.

Next, we briefly describe the simplicial version of Bredon-Illman cohomology with local coefficients as introduced in [16].

Let X be a G -simplicial set and M an equivariant local coefficients on X . For each equivariant n -simplex $\sigma : G/H \times \Delta[n] \rightarrow X$, we associate an equivariant 0-simplex $\sigma_H : G/H \times \Delta[0] \rightarrow X$ given by

$$\sigma_H = \sigma \circ (id \times \delta_{(1,2,\dots,n)}),$$

where $\delta_{(1,2,\dots,n)}$ is the composition

$$\delta_{(1,2,\dots,n)} : \Delta[0] \xrightarrow{\delta_1} \Delta[1] \xrightarrow{\delta_2} \dots \xrightarrow{\delta_n} \Delta[n].$$

The j -th face of σ is an equivariant $(n-1)$ -simplex of type H , denoted by $\sigma^{(j)}$, and is defined by

$$\sigma^{(j)} = \sigma \circ (id \times \delta_j), \quad 0 \leq j \leq n.$$

Remark 3.5. Note that $\sigma_H^{(j)} = \sigma_H$ for $j > 0$, and

$$\sigma_H^{(0)} = \sigma \circ (id \times \delta_{(0,2,\dots,n)}).$$

Let $C_G^n(X; M)$ be the group of all functions f defined on equivariant n -simplexes $\sigma : G/H \times \Delta[n] \rightarrow X$ such that $f(\sigma) \in M(\sigma_H)$ with $f(\sigma) = 0$, if σ is degenerate. We have a morphism $\sigma_* = [id, \alpha]$ in ΠX from σ_H to $\sigma_H^{(0)}$ induced by σ , where $\alpha : G/H \times \Delta[1] \rightarrow X$ is given by $\alpha = \sigma \circ (id \times \delta_{(2,\dots,n)})$. Define a homomorphism

$$\delta : C_G^n(X; M) \rightarrow C_G^{n+1}(X; M)$$

$$f \mapsto \delta f$$

where for any equivariant $(n+1)$ -simplex σ of type H ,

$$(\delta f)(\sigma) = M(\sigma_*)(f(\sigma^{(0)})) + \sum_{j=1}^{n+1} (-1)^j f(\sigma^{(j)}).$$

A routine verification shows that $\delta \circ \delta = 0$. Thus $\{C_G^*(X; M), \delta\}$ is a cochain complex. We are interested in a subcomplex of this cochain complex as defined below.

Let $\eta : G/H \times \Delta[n] \rightarrow X$ and $\tau : G/K \times \Delta[n] \rightarrow X$ be two equivariant n -simplexes. Suppose there exists a G -map $\hat{g} : G/H \rightarrow G/K$, $g^{-1}Hg \subseteq K$, such that $\tau \circ (\hat{g} \times id) = \eta$. Then η and τ are said to be compatible under \hat{g} . Observe that if η and τ are compatible as described above then η is degenerate if and only if τ is degenerate. Moreover notice that in this case, we have a morphism $[\hat{g}, k] : \eta_H \rightarrow \tau_K$ in ΠX , where $k = \eta_H \circ (id \times \sigma_0)$, where $\sigma_0 : \Delta[1] \rightarrow \Delta[0]$ is the map as described in Section 2. Let us denote this induced morphism by g_* .

Definition 3.6. We define $S_G^n(X; M)$ to be the subgroup of $C_G^n(X; M)$ consisting of all functions f such that if η and τ are equivariant n -simplexes in X which are compatible under \hat{g} then $f(\eta) = M(g_*)(f(\tau))$.

If $f \in S_G^n(X; M)$ then one can verify that $\delta f \in S_G^{n+1}(X; M)$.

Thus we have a cochain complex $S_G(X; M) = \{S_G^n(X; M), \delta\}$.

Definition 3.7. Let X be a G -simplicial set with equivariant local coefficients M on it. Then the n -th Bredon-Illman cohomology of X with local coefficients M is defined by

$$H_G^n(X; M) = H^n(S_G(X; M)).$$

4 Equivariant Eilenberg Theorem

Let X be a one vertex Kan complex. For any $x \in X_1$, we denote by $[x]$ the element of $\pi = \pi_1(X, v)$ represented by the 1-simplex x where v is the unique vertex of X . Recall that ([6], [8]) the universal covering complex \tilde{X} of X is defined as follows:

$$\tilde{X}_n = \pi \times X_n$$

with the face maps

$$\partial_i(\gamma, x) = (\gamma, \partial_i x), \quad 0 < i \leq n, \quad x \in X_n, \quad \gamma \in \pi$$

$$\partial_0(\gamma, x) = ([\partial_{(2,3,\dots,n)}x]\gamma, \partial_0 x),$$

where $\partial_{(2,3,\dots,n)}x = \partial_2 \partial_3 \cdots \partial_n x$. The degeneracy maps are

$$s_i(\gamma, x) = (\gamma, s_i x) \quad 0 \leq i \leq n.$$

Then $p : \tilde{X} \rightarrow X$, p being the first projection, has the usual properties of universal covering. Any map $f : X \rightarrow Y$ of such complexes induces a map $\tilde{f} : \tilde{X} \rightarrow \tilde{Y}$ by $\tilde{f}(\gamma, x) = (f_*(\gamma), f(x))$, where $f_* : \pi_1(X) \rightarrow \pi_1(Y)$ is the homomorphism of fundamental groups induced by f .

Remark 4.1. We note that given any two 0-simplexes $x_1 = (\gamma_1, v)$ and $x_2 = (\gamma_2, v)$ in \tilde{X} , there is a unique homotopy class of 1-simplexes ω such that $\partial_1 \omega = x_1$, $\partial_0 \omega = x_2$, as \tilde{X} is simply connected. We may represent this class by $\omega = (\gamma_1, \omega_2 \omega_1^{-1})$ where ω_i represents γ_i , $i = 1, 2$.

The fundamental group $\pi_1(X)$ operates on \tilde{X} freely by

$$(\sigma, (\gamma, x)) \mapsto (\gamma \sigma^{-1}, x).$$

This action is natural with respect to maps of complexes and an analogue of Eilenberg Theorem holds. The purpose of this section is to prove an equivariant version of this result. We define a contravariant functor from the category of canonical orbits to the category of one vertex Kan complexes as follows.

Let X be a one vertex G -Kan complex. We denote the G -fixed vertex by v . For any subgroup H of G , let

$$p_H : \widetilde{X^H} \rightarrow X^H$$

be the universal cover of X^H . The left translation $a : X^K \rightarrow X^H$, corresponding to a G -map $\hat{a} : G/H \rightarrow G/K$, $a^{-1}Ha \subseteq K$, induces a simplicial map $\tilde{a} : \widetilde{X^K} \rightarrow \widetilde{X^H}$ such that $p_H \circ \tilde{a} = a \circ p_K$. This defines an O_G -Kan complex \tilde{X} by setting $\tilde{X}(G/H) = \widetilde{X^H}$ and $\tilde{X}(\hat{a}) = \tilde{a}$. This is called the universal O_G -covering complex of X . This is simplicial analogue of O_G -covering space as introduced in [13]. We refer [10] for a more general version, called 'universal covering functor'. For any subgroup H , let $\tilde{v}^H \in \widetilde{X^H}$ denote the zero simplex (e_H, v) , where e_H is the identity element of $\pi_1 X(G/H) = \pi_1(X^H, v)$. Note that the map \tilde{a} induced by $a : X^K \rightarrow X^H$ maps \tilde{v}^K to \tilde{v}^H .

The natural actions of $\pi_1 X(G/H) = \pi_1(X^H, v)$ on $\tilde{X}(G/H) = \widetilde{X^H}$ as H varies over subgroups of G , define an action of the O_G -group $\pi_1 X$ on \tilde{X} .

Suppose M is an equivariant local coefficients on X . We have an abelian O_G -group M_0 associated to M as described below.

For any subgroup H of G , let v_H be the object of type H in ΠX defined by

$$v_H : G/H \times \Delta[0] \rightarrow X,$$

$$(eH, \Delta_0) \mapsto v.$$

Then for any morphism $\hat{g} : G/H \rightarrow G/K$ in O_G , given by a subconjugacy relation $g^{-1}Hg \subseteq K$, we have a morphism $[\hat{g}, k] : v_H \rightarrow v_K$ in ΠX , where $k : G/H \times \Delta[1] \rightarrow X$ is given by $k(eH, \Delta_1) = s_0 v$. Define $M_0 : O_G \rightarrow Ab$ by $M_0(G/H) = M(v_H)$ and $M_0(\hat{g}) = M[\hat{g}, k]$. The abelian O_G -group M_0 comes equipped with a natural action of the O_G group $\pi_1 X$ as described below.

Let $\alpha = [\phi'] \in \pi_1 X(G/H) = \pi_1(X^H, v)$. Then the morphism $[id, \phi] : v_H \rightarrow v_H$ where $\phi(eH, \Delta_1) = \phi'(\Delta_1)$, is an equivalence in the category ΠX . This yields a group homomorphism

$$b : \pi_1(X^H, v) \rightarrow Aut_{\Pi X}(v_H), \quad \alpha = [\phi'] \mapsto b(\alpha) = [id, \phi].$$

We remark that the composition of the fundamental group $\pi_1(X^H, v)$ coincides with the morphism composition of ΠX , contrary to the usual topological composition in the fundamental group. The composition of the map b with the group homomorphism $Aut_{\Pi X}(v_H) \rightarrow Aut_{Ab}(M(v_H))$ which sends $\alpha \in Aut_{\Pi X}(v_H)$ to $[M(\alpha)]^{-1}$ defines the action of $\pi_1(X^H, v)$ on $M_0(G/H)$. It is routine to check that this action is natural with respect to morphism of O_G .

We define cohomology groups of \tilde{X} with coefficients in M_0 using invariance of actions of $\pi_1 X$ on \tilde{X} and M_0 as follows.

We have a chain complex $\{\underline{C}_n(\tilde{X}), \underline{\partial}_n\}$ in the abelian category \mathcal{C}_G defined by

$$\underline{C}_n(\tilde{X})(G/H) := C_n(\tilde{X}^H; \mathbb{Z})$$

for every object G/H and for every morphism $\hat{a} : G/H \rightarrow G/K$, $a^{-1}Ha \subseteq K$, $\underline{C}_n(\hat{a}) := \tilde{a}_\# : C_n(\tilde{X}^K; \mathbb{Z}) \rightarrow C_n(\tilde{X}^H; \mathbb{Z})$, where $C_n(\tilde{X}^H; \mathbb{Z})$ denotes the free abelian group generated by the non-degenerate n -simplexes of \tilde{X}^H . The boundary map $\underline{\partial}_n : \underline{C}_n(\tilde{X}) \rightarrow \underline{C}_{n-1}(\tilde{X})$ is the

natural transformation, defined by $\underline{\partial}_n(G/H) := \partial_n$ where $\partial_n : C_n(\tilde{X}^H; \mathbb{Z}) \rightarrow C_{n-1}(\tilde{X}^H; \mathbb{Z})$ is the ordinary boundary map. Note that $\underline{\pi}X$ acts on the chain complex $\{\underline{C}_n(\tilde{X}), \underline{\partial}_n\}$ via its action on \tilde{X} . We now form the cochain complex $\{Hom_{\underline{\pi}X}(\underline{C}_n(\tilde{X}), M_0), \delta^n\}$, where $Hom_{\underline{\pi}X}(\underline{C}_n(\tilde{X}), M_0)$ consists of all natural transformations $\underline{C}_n(\tilde{X}) \rightarrow M_0$ respecting the actions of $\underline{\pi}X$ and $\delta^n f = f \circ \partial_{n+1}$. Then the n^{th} equivariant cohomology of \tilde{X} with coefficients in M_0 is defined by

$$H_{\underline{\pi}X, G}^n(\tilde{X}; M_0) := H_n(Hom_{\underline{\pi}X}(\underline{C}_\#(\tilde{X}), M_0)).$$

Theorem 4.2. *Let X be a one vertex G -Kan complex and M be an equivariant local coefficients on X . Then*

$$H_G^n(X; M) \cong H_{\underline{\pi}X, G}^n(\tilde{X}; M_0).$$

Proof. Recall that for any two 0-simplexes $x, y \in \tilde{X}^H$ of the universal cover of the H -fixed point complex X^H , there is a unique homotopy class of 1-simplexes ω with $\partial_1 \omega = x$ and $\partial_0 \omega = y$. Let us denote this class by $\xi_H(x, y)$. In particular, if $x = \tilde{v}^H$, then we shall write $\tilde{\xi}(\tilde{v}^H, y)$ simply by $\tilde{\xi}_H(y)$. Upon projecting $\tilde{\xi}_H(x, y)$ via p_H we get an element $\xi_H(x, y) \in \pi_1(X^H, v)$. By Remark 3.3, $\xi_H(x, y)$ corresponds to an automorphism $b\xi_H(x, y)$ of v_H in ΠX . As before $\xi_H(x, y)$ will be denoted by $\xi_H(y)$ when $x = \tilde{v}^H$. Define a map

$$\phi : S_G^n(X; M) \longrightarrow Hom_{\underline{\pi}X}(\underline{C}_n(\tilde{X}), M_0)$$

as follows. Let $f \in S_G^n(X; M)$ and y be a non-degenerate n -simplex in \tilde{X}^H . Let σ be the equivariant n -simplex of type H in X such that $\sigma' = p_H \circ \overline{y}$, where $\overline{y} : \Delta[n] \rightarrow \tilde{X}^H$ is the simplicial map with $\overline{y}(\Delta_n) = y$. Then $\phi(f) \in Hom_{\underline{\pi}X}(\underline{C}_n(\tilde{X}), M_0)$ is given by

$$\phi(f)(G/H)(y) = M(b\xi_H(\partial_{(1,2,\dots,n)}y))f(\sigma).$$

Recall that $f(\sigma) \in M(\sigma_H)$ and σ_H in this case coincides with v_H .

We check that $\phi(f)(G/H)$ is equivariant with respect to the respective actions of $\underline{\pi}X(G/H)$ on $\underline{C}_n(\tilde{X})(G/H)$ and on $M_0(G/H)$. Let $u \in \underline{\pi}X(G/H)$, $y \in \tilde{X}_n^H$ and σ be the equivariant n -simplex determined by y as above. Then

$$\phi(f)(G/H)(uy) = M(b\xi_H(\partial_{(1,2,\dots,n)}uy))f(\tau),$$

where $\tau' = p_H \circ \overline{uy}$. By the definition of the action of $\underline{\pi}X(G/H)$ on $C_n(\tilde{X}^H; \mathbb{Z})$, we have $p_H \circ \overline{uy} = p_H \circ \overline{y}$, hence $\tau' = \sigma'$. It follows that

$$\phi(f)(G/H)(uy) = M(b\xi_H(\partial_{(1,2,\dots,n)}uy))f(\sigma).$$

Now write $\tilde{\xi}_H(\partial_{(1,2,\dots,n)}uy)$ as a composition

$$\tilde{\xi}_H(u\tilde{v}^H, \partial_{(1,2,\dots,n)}uy) \circ \tilde{\xi}_H(u\tilde{v}^H)$$

of morphisms in the fundamental groupoid of \tilde{X}^H . Observe that by Remark 4.1, $\xi_H(u\tilde{v}^H) = u^{-1}$ and $\xi_H(u\tilde{v}^H, \partial_{(1,2,\dots,n)}uy) = \xi_H(\partial_{(1,2,\dots,n)}uy)$. Therefore

$$M(b\xi_H(\partial_{(1,2,\dots,n)}uy)) = M(bu)^{-1} \circ M(b\xi_H(\partial_{(1,2,\dots,n)}uy)).$$

Thus $\phi(f)(G/H)(uy) = M(bu)^{-1}\phi(f)(G/H)(y)$. It follows from the definition of the action of $\pi X(G/H)$ on $M_0(G/H)$ that $\phi(f)(G/H)$ is equivariant.

To check that $\phi(f)(G/H) : \underline{\mathcal{C}}_n(\tilde{X}) \rightarrow M_0$ is natural, we have to show that

$$M_0(\hat{g}) \circ \phi(f)(G/K) = \phi(f)(G/H) \circ \tilde{g}_\#$$

whenever $g^{-1}Hg \subseteq K$. Recall that by definition of M_0 , $M_0(\hat{g}) = M(v_H \xrightarrow{[\hat{g}, k]} v_K)$, where $k : G/H \times \Delta[1] \rightarrow X$ is given by $k(eH, \Delta_1) = s_0v$. Let $y \in \tilde{X}_n^K$ and $g^{-1}Hg \subseteq K$. Let τ be an equivariant n -simplex of type K in X such that $\tau' = p_K \circ \overline{y}$. Then

$$\begin{aligned} & M_0(\hat{g}) \circ \phi(f)(G/K)(y) \\ &= M(v_H \xrightarrow{[\hat{g}, k]} v_K) \circ M(b\xi_K(\partial_{(1,2,\dots,n)}y))f(\tau) \\ &= M(v_H \xrightarrow{[\hat{g}, k]} v_K) \circ M([id_{G/K}, \omega])f(\tau) \\ &= M([id_{G/H}, \omega] \circ [\hat{g}, k])f(\tau), \end{aligned}$$

where as in Remark 3.3, ω is the equivariant 1-simplex of type K in X such that ω' represents $\xi_K(\partial_{(1,2,\dots,n)}y)$. On the other hand,

$$\begin{aligned} & \phi(f)(G/H) \circ \tilde{g}_\#(y) \\ &= \phi(f)(G/H)(\tilde{g}y) \\ &= M(b\xi_H(\partial_{(1,2,\dots,n)}\tilde{g}y))f(\sigma) \end{aligned}$$

where $\sigma' = p_H \circ \overline{\tilde{g}y} = p_H \circ \tilde{g} \circ \overline{y} = g \circ p_K \circ \overline{y} = g \circ \tau'$. In particular, σ and τ are compatible n -simplexes. Thus

$$\phi(f)(G/H) \circ \tilde{g}_\#(y) = M(b\xi_H(\partial_{(1,2,\dots,n)}\tilde{g}y)) \circ M(g_*)f(\tau).$$

Note that v is the only vertex in X which is G -fixed and hence g_* is a morphism from v_H to v_K and is given by $[\hat{g}, k]$ where $k = v_H \circ (id_{G/H} \times \sigma_0)$. Now observe that $\xi_H(\partial_{(1,2,\dots,n)}\tilde{g}y) = \xi_H(\tilde{g}\partial_{(1,2,\dots,n)}y)$ can be represented by $g\omega'$. As a consequence we may write

$$b\xi_H(\partial_{(1,2,\dots,n)}\tilde{g}y) = [id_{G/H}, \omega \circ (\hat{g} \times id_{\Delta[1]})].$$

Therefore

$$\begin{aligned} & \phi(f)(G/H) \circ \tilde{g}_\#(y) \\ &= M([id_{G/H}, \omega \circ (\hat{g} \times id_{\Delta[1]})]) \circ M([\hat{g}, k])f(\tau) \\ &= M([\hat{g}, k] \circ [id_{G/H}, \omega \circ (\hat{g} \times id_{\Delta[1]})])f(\tau). \end{aligned}$$

From the definition of composition of morphism in ΠX , we have

$$[id_{G/K}, \omega] \circ [\hat{g}, k] = [\hat{g}, k] \circ [id_{G/H}, \omega \circ (\hat{g} \times id_{\Delta[1]})].$$

Thus $\phi(f)$ is natural.

Next we check that ϕ is a chain map. Let $f \in S_G^n(X; M)$, $y \in \widetilde{X_{n+1}^H}$. Let σ denotes the equivariant $(n+1)$ -simplex of type H corresponding to y as described before. Then

$$\begin{aligned} & \phi(\delta f)(G/H)(y) \\ &= M(b\xi_H(\partial_{(1,2,\dots,n+1)}y))(\delta f)(\sigma) \\ &= M(b\xi_H(\partial_{(1,2,\dots,n+1)}y)) \left\{ M(\sigma_*)f(\sigma^{(0)}) + \sum_{j=1}^{n+1} (-1)^j f(\sigma^{(j)}) \right\}. \end{aligned}$$

On the other hand,

$$\begin{aligned} & \delta\phi(f)(G/H)(y) \\ &= \sum_{i=0}^{n+1} (-1)^i \phi(f)(G/H)(\partial_i y) \\ &= \sum_{i=0}^{n+1} (-1)^i M(b\xi_H(\partial_{(1,2,\dots,n)}\partial_i y))f(\sigma^{(i)}) \\ &= M(b\xi_H(\partial_{(0,2,\dots,n+1)}y))f(\sigma^{(0)}) + \sum_{i=1}^{n+1} (-1)^i M(b\xi_H(\partial_{(1,2,\dots,n+1)}y))f(\sigma^{(i)}). \end{aligned}$$

Now note that since $\widetilde{X^H}$ is simply connected the morphism $\xi_H(\partial_{(0,2,\dots,n+1)}y)$ in πX^H can be factored as

$$\xi_H(\partial_{(1,2,\dots,n+1)}y, \partial_{(0,2,\dots,n+1)}y) \circ \xi_H(\partial_{(1,2,\dots,n+1)}y)$$

and $b\xi_H(\partial_{(1,2,\dots,n+1)}y, \partial_{(0,2,\dots,n+1)}y)$ is precisely the morphism σ_* . Therefore

$$b\xi_H(\partial_{(0,2,\dots,n+1)}y) = \sigma_* \circ b\xi_H(\partial_{(1,2,\dots,n+1)}y).$$

Hence $\phi(\delta f) = \delta\phi(f)$.

To show that ϕ is a chain isomorphism define a map

$$\psi : Hom_{\underline{X}}(\underline{C}_n(\tilde{X}), M_0) \rightarrow C_G^n(X; M)$$

as follows. Let $f \in Hom_{\underline{X}}(\underline{C}_n(\tilde{X}), M_0)$ and σ be a non-degenerate equivariant n -simplex in X of type H . Choose an n -simplex y in $\widetilde{X^H}$ such that $p_H(y) = \sigma(eH, \Delta_n)$. Then $\psi(f)$ is given by

$$\psi(f)(\sigma) = M(b\xi_H(\partial_{(1,2,\dots,n)}y))^{-1}f(G/H)(y).$$

Suppose z is another n -simplex in $\widetilde{X^H}$ such that $p_H(z) = \sigma(eH, \Delta_n)$. Since $\pi_1(X^H, v)$ acts transitively on each fibre of $p_H : \widetilde{X^H} \rightarrow X^H$, there exists an element $u \in \pi_1(X^H, v)$ such that $uy = z$ and hence $u\partial_{(1,2,\dots,n)}y = \partial_{(1,2,\dots,n)}z$. Thus

$$\begin{aligned} & M(b\xi_H(\partial_{(1,2,\dots,n)}z))^{-1}f(G/H)(z) \\ &= M(b\xi_H(\partial_{(1,2,\dots,n)}uy))^{-1}f(G/H)(uy) \\ &= M(b\xi_H(\partial_{(1,2,\dots,n)}y))^{-1}M(bu)^{-1}f(G/H)(y) \\ &= M(b\xi_H(\partial_{(1,2,\dots,n)}y))^{-1}f(G/H)(y). \end{aligned}$$

The last equality follows from the observation

$$M(b\xi_H(\partial_{(1,2,\dots,n)}uy)) = M(bu)^{-1} \circ M(b\xi_H(\partial_{(1,2,\dots,n)}y)),$$

while showing ϕ takes any cocycle in $S_G^n(X; M)$ into $\text{Hom}_{\underline{\mathbb{A}}X}(\underline{C}_n(\tilde{X}), M_0)$, in the first part of the proof. Thus the map ψ is well defined.

We claim that $\psi(f) \in S_G^n(X; M)$ for any $f \in \text{Hom}_{\underline{\mathbb{A}}X}(\underline{C}_n(\tilde{X}), M_0)$. Let $a^{-1}Ha \subseteq K$ and $\sigma : G/H \times \Delta[n] \rightarrow X$ and $\eta : G/K \times \Delta[n] \rightarrow X$ be equivariant n -simplexes such that $\eta \circ (\hat{a} \times id) = \sigma$ so that they are compatible. We need to show that $\psi(f)(\sigma) = M(a_*)\psi(f)(\eta)$. Let $y \in \widehat{X^K}$ be such that $p_K(y) = \eta(eK, \Delta_n)$. Then the n -simplex $\tilde{a}y \in \widehat{X_n^H}$ is such that

$$p_H(\tilde{a}y) = ap_K(y) = a\eta(eK, \Delta_n) = \eta(aK, \Delta_n) = \sigma(eH, \Delta_n).$$

By our choice, we have

$$\psi(f)(\eta) = M(b\xi_K(\partial_{(1,2,\dots,n)}y))^{-1}f(G/K)(y)$$

and

$$\psi(f)(\sigma) = M(b\xi_H(\partial_{(1,2,\dots,n)}\tilde{a}y))^{-1}f(G/H)(\tilde{a}y).$$

Since $f : \underline{C}_n(\tilde{X}) \rightarrow M_0$ is natural, we have

$$f(G/H)(\tilde{a}y) = M_0(\hat{a})(f(G/K)(y)).$$

In the first part of the proof we have observed that

$$a_* \circ b\xi_H(\partial_{(1,2,\dots,n)}\tilde{a}y) = b\xi_K(\partial_{(1,2,\dots,n)}y) \circ a_*.$$

Moreover, recall that $M_0(\hat{a}) = M(a_*)$. Therefore

$$\begin{aligned} & M(a_*)\psi(f)(\eta) \\ &= M(a_*)M(b\xi_K(\partial_{(1,2,\dots,n)}y))^{-1}f(G/K)(y) \\ &= M(b\xi_K(\partial_{(1,2,\dots,n)}y)^{-1} \circ a_*)f(G/K)(y) \\ &= M(a_* \circ b\xi_H(\partial_{(1,2,\dots,n)}\tilde{a}y))f(G/K)(y) \\ &= M(b\xi_H(\partial_{(1,2,\dots,n)}\tilde{a}y))^{-1}M(a_*)f(G/K)(y) \\ &= M(b\xi_H(\partial_{(1,2,\dots,n)}\tilde{a}y))^{-1}M_0(\hat{a})f(G/K)(y) \\ &= \psi(f)(\sigma). \end{aligned}$$

It is routine to check that ψ is the inverse of ϕ . This completes the proof of the theorem. \square

5 Equivariant Serre spectral sequence

The aim of this last section is to derive a version of Serre spectral sequence. To do this we give an alternative description of equivariant simplicial cohomology with local coefficients in terms of cohomology of small categories.

Let G be a discrete group and X a G -Kan complex. Then we have a category $\Delta_G(X)$ described as follows. Its objects are G -simplicial maps $\sigma : G/H \times \Delta[n] \rightarrow X$ and a morphism from $\sigma : G/H \times \Delta[n] \rightarrow X$ to $\tau : G/K \times \Delta[m] \rightarrow X$ is a pair (\hat{g}, α) where $\hat{g} : G/H \rightarrow G/K$ is a G -map and $\alpha : \Delta[n] \rightarrow \Delta[m]$ is a simplicial map such that $\tau \circ (\hat{g} \times \alpha) = \sigma$. There is a canonical

functor $v_X : \Delta_G(X) \rightarrow \Pi(X)$ which sends $\sigma : G/H \times \Delta[n] \rightarrow X$ to $\sigma_H = \sigma \circ (id \times \delta_{(1,2,\dots,n)})$. For a morphism (\hat{g}, α) in $\Delta_G(X)$, $v_X(\hat{g}, \alpha) : \sigma_H \rightarrow \tau_K$ is the morphism $[\hat{g}, \phi]$ in $\Pi(X)$ where $\phi : G/H \times \Delta[1] \rightarrow X$ is an equivariant 1-simplex of type H obtained as follows. Suppose $\tau \circ (id \times \delta_{(1,\dots,\widehat{\alpha(0)},\dots,m)}) (eK, \Delta_1) = \omega \in X^K$. Let x be a 2-simplex in X^K determined by the compatible pair of 1-simplices $(\cdot, s_1 \partial_1 \omega, \omega)$. Then ϕ is given by $\phi(eH, \Delta_1) = g(\partial_0 x)$.

If X is any G -simplicial set then we define $\Delta_G(X) = \Delta_G(S|X|)$.

For a small category \mathcal{C} , let $\mathcal{Ab}(\mathcal{C})$ be the category of all contravariant functors from \mathcal{C} to \mathcal{Ab} with morphisms natural transformations of functors.

Definition 5.1. A functor $M \in \mathcal{Ab}(\Delta_G(X))$ is said to be G -local if

$$M = v_X^* M' = M' \circ v_X$$

for some $M' \in \mathcal{Ab}(\Pi(X))$. For a G -local coefficients M , the equivariant cohomology of X with coefficients M is defined to be

$$h_G^*(X; M) := H^*(\Delta_G(X); M),$$

where the right hand side denotes the cohomology of the category $\Delta_G(X)$, in the sense of [17].

Theorem 5.2. Let X be a G -simplicial set and M be an equivariant local coefficients on X . Then there is an isomorphism

$$H_G^*(X; M) \cong h_G^*(X; M).$$

(On the right we identify M with $v_X^*(M)$).

Proof. Let $N\mathcal{C}$ denote the nerve of a small category \mathcal{C} . Then as in ([12]) we let \tilde{X} be the bisimplicial set whose (p, q) simplices are triples (u, α, σ) where

$$\begin{aligned} u &= ([n_0] \xrightarrow{u_1} [n_1] \rightarrow \dots \xrightarrow{u_p} [n_p]) \in N_p(\Delta) \\ \alpha &= (G/H_0 \xrightarrow{\alpha_1} G/H_1 \rightarrow \dots \xrightarrow{\alpha_q} G/H_q) \in N_q(O_G) \\ \sigma &: G/H_q \times \Delta[n_p] \rightarrow X \text{ is a } G\text{-simplicial map.} \end{aligned}$$

The face and degeneracy maps on \tilde{X} are induced from those on $N(\Delta)$ and $N(O_G)$. Then

$$\text{diagonal}(\tilde{X}) \cong N(\Delta_G(X)).$$

To every $(u, \alpha, \sigma) \in \tilde{X}^{p,q}$ associate a G -simplicial map,

$$\bar{\sigma} = \sigma \circ (\alpha_q \circ \dots \circ \alpha_1 \times u_p \circ \dots \circ u_1) : G/H_0 \times \Delta[n_0] \rightarrow X.$$

Let $C^{p,q}(X; M)$ denote the set of all functions on $\tilde{X}^{p,q}$ which sends an element (u, α, σ) of $\tilde{X}^{p,q}$ to an element of $M(v_X(\bar{\sigma}))$. It follows quite easily that $C^{p,q}(X; M)$ is a bicomplex with obvious differentials d_h and d_v induced from the face maps of \tilde{X} . Denote the total complex of $C^{\bullet\bullet}(X, M)$ by $\text{Tot } C^{\bullet\bullet}(X; M)$. Let $\text{diag } C^{\bullet\bullet}(X; M)$ be the cochain complex whose p^{th} group is $C^{p,p}(X; M)$ and differential is $d_h d_v$. Then by a result of Dold and Puppe ([4]) we have

$$H^n(\text{Tot } C^{\bullet\bullet}(X; M)) \cong H^n(\text{diag } C^{\bullet\bullet}(X; M)).$$

Now $C^{p,p}(X; M)$ can be interpreted as the set of all functions on $N(\Delta_G(X))$ which sends a p -simplex $(\tau_0 \rightarrow \tau_1 \rightarrow \cdots \rightarrow \tau_p)$ to an element of $M(v_X(\tau_0))$ and the differential on $C^{p,p}(X; M)$ is just the differential induced from the face maps of $N_p(\Delta_G(X))$. Hence,

$$H^n(\text{diag } C^{\bullet\bullet}(X; M)) \cong H^n(\Delta_G(X); v_X^* M) = h_G^n(X; M).$$

Recall that the spectral sequence associated to the p -filtration of the bicomplex $C^{\bullet\bullet}(X; M)$ converges to the cohomology of the total complex. Now proceeding as in [15], we may compute the E_1 term of the spectral sequence. It turns out that

$$\begin{aligned} E_1^{p,q} = H^q(C^{p,\bullet}(X; M)) &\cong \begin{cases} \Pi_{u \in N_p(\Delta)} S_G^{n(u)}(X; M) & \text{if } q = 0 \\ 0 & \text{if } q > 0, \end{cases} \end{aligned}$$

where $S_G^{n(u)}(X; M)$ is a copy of $S_G^{n_p}(X; M)$ for every $u = ([n_0] \rightarrow \cdots \rightarrow [n_p])$. (See [15] for details.) Thus,

$$\begin{aligned} H^p(\text{Tot } C^{\bullet\bullet}(X; M)) &\cong H^p(\Pi_{u \in N(\Delta)} S_G^{n(u)}(X; M)) \\ &\cong H^p(\Delta^{op}, S_G^\bullet(X; M)) \end{aligned}$$

where $S_G^\bullet(X; M)$ is the cosimplicial group which takes $[n]$ to $S_G^n(X; M)$ with obvious face and degeneracy maps induced from those on Δ . Then we know that ([12]),

$$H^p(\Delta^{op}; S_G^\bullet(X; M)) \cong H^p(S_G^\bullet(X; M)).$$

Hence,

$$H^p(\text{Tot } C^{\bullet\bullet}(X; M)) \cong H_G^p(X; M).$$

□

We are now in a position to derive the required spectral sequence.

Let X, Y be G -simplicial sets and $f : Y \rightarrow X$ be a G -Kan fibration and M a G -local coefficients on Y . For $q \geq 0$ we have a contravariant functor

$$h_G^q(f, M) : \Delta_G(X) \rightarrow \mathcal{A}b$$

as follows. For an object $\sigma : G/H \times \Delta[n] \rightarrow X$ of $\Delta_G(X)$, let $\sigma^*(Y)$ be the G -simplicial set obtained by pulling back f along σ and define

$$h_G^q(f, M)(\sigma) := h_G^q(\sigma^*(Y); \tilde{\sigma}^* M),$$

where $\tilde{\sigma} : \sigma^*(Y) \rightarrow Y$ is the canonical map. We claim that $h_G^q(f, M)$ factors through v_X yielding a G -local coefficients on X . To see this, first note that the following result holds ([12], the proof of the Theorem 2.3).

Theorem 5.3. *Let $f : Y \rightarrow X$ be a weak equivalence in GS . Then for any G -local coefficients M on X , f induces an isomorphism*

$$h_G^*(X; M) \cong h_G^*(Y; f^* M).$$

Recall that $f : Y \longrightarrow X$ is a weak equivalence in GS if and only if $f^H : Y^H \longrightarrow X^H$ is a weak equivalence in \mathcal{S} . Next note that if $u : \Delta[m] \longrightarrow \Delta[n]$ is a simplicial map then the map $\sigma(id \times u)^*(Y) \longrightarrow \sigma^*(Y)$ covering $id \times u : G/K \times \Delta[m] \longrightarrow G/K \times \Delta[n]$ is a weak equivalence in GS , as f is a G -Kan fibration. The claim now follows from the above result.

Theorem 5.4. *For any G -Kan fibration $f : Y \rightarrow X$ and a G -local coefficients M on Y , there is a natural spectral sequence with E_2 -term $E_2^{p,q} = H_G^p(X; h_G^p(f, M))$ converging to $H_G^{p+q}(Y; M)$.*

Proof. The proof is parallel to the proof of Theorem 3.2, [12]. We only mention the essential steps. The G -Kan fibration $f : Y \rightarrow X$ induces a functor $\Delta_G(f) : \Delta_G(Y) \rightarrow \Delta_G(X)$ and we have a Grothendieck spectral sequence [18]

$$H^p(\Delta_G(X); h^q(\Delta_G(f)/-; M)) \Rightarrow H^{p+q}(\Delta_G(Y); M).$$

It is enough to show that the two contravariant functors $h^q(\Delta_G(Y)/-; M)$ and $h_G^q(f, M)$ from $\Delta_G(X)$ to $\mathcal{A}b$ are equivalent. For an object σ of $\Delta_G(X)$, $\Delta_G(f)/\sigma$ is the comma category. Objects of $\Delta_G(f)/\sigma$ are pairs (τ, u) where $\tau \in Ob(\Delta_G(Y))$ and $u : \Delta_G(f)(\tau) \rightarrow \sigma$ is a map in $\Delta_G(X)$. Morphisms from (τ, u) to (τ', u') are maps $\alpha : \tau \rightarrow \tau'$ such that $u' \Delta_G(f)(\alpha) = u$. A direct computation shows that there is a canonical equivalence of the categories

$$\Delta_G(f)/\sigma \cong \Delta_G(\sigma^*(Y)),$$

which is natural in σ . Hence we have natural isomorphism of functors

$$h^q(\Delta_G(\sigma^*(Y); \tilde{\sigma}^* M)) \cong h^q(\Delta_G(f)/\sigma; M).$$

The result now follows from Theorem 5.2. □

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